Lubin-Tate Theorem and Construction of Morava E-Theory

Ze, FAN

Wesleyan University

August 6, 2025

The Category of Formal Group Laws

Definition

Let R be a commutative ring. A *formal group law* over the ring R is a power series $F \in R[[x, y]]$ satisfying the following three properties:

(a) $F(x, y) = F(y, x)$,

(b) $F(x, 0) = x$, and $F(0, y) = y$,

(c) $F(x, F(y, z)) = F(F(x, y), z)$ in $R[[x, y, z]]$.

Definition

For $F, G \in \mathrm{Fgl}\, R$, a morphism $f : F \to G$ is a power series $f \in R[[x]]$ such that $f(0) = 0$ and

$$f(F(x, y)) = G(f(x), f(y)).$$

Examples

For any ring R:

- The additive formal group law $F(x, y) = x + y$
- The multiplicative formal group law $F(x, y) = x + y + rxy$, for $r \in R$.

For $R = \mathbb{F}_{p^n}$:

- The Honda formal group law $H_n(x, y)$, such that

$$[p]_{H_n}(x) := \underbrace{x +_{H_n} x +_{H_n} \cdots +_{H_n} x}_{p \text{ times}}$$
$$= F(\cdots F(F(x, x), x) \cdots , x) = x^{p^n}.$$

Why Formal Group Laws are Important?

Recall that the Chern classes c_k for $k \in \mathbb{Z}_+$ is an assignment, assigning each complex vector bundle $V \to X$ to a singular cohomology class $c_k(V) \in H^{2k}(X, \mathbb{Z})$.
The total Chern class given by

$$c(V) := 1 + c_1(V) + c_2(V) + \cdots$$

satisfies

- $c(f^*V) = f^*c(V)$ for continuous $f : X \to Y$,
- $c(V \oplus W) = c(V) \sqcup c(W)$,
- $c(\mathcal{O}(1)) = 1 + t$, where $\mathcal{O}(1) \to \mathbb{C}P^\infty$ is the tautological line bundle, and $t \in H^2(\mathbb{C}P^\infty, \mathbb{Z}) \cong \mathbb{Z}$ is a generator.

The three properties uniquely determines the total Chern class.

Complex Orientable Cohomology Theories

Definition

A *complex orientation* for a multiplicative cohomology theory $E^*(-)$ is a choice of an element $x \in E^2(\mathbb{C}P^\infty)$ which under the map

$$E^2(\mathbb{C}P^\infty) \to E^2(\mathbb{C}P^1) \cong E^2(S^2) \cong E^0(\mathrm{pt})$$

restricts to the unit $1 \in E^0(\mathrm{pt})$.

We can then define a "generalized version" of first Chern class on E by setting $c_1(\mathcal{O}(1)) = x$ and $c_1(L) = f^*(x) \in E^2(X)$ for general line bundle $L \to X$, where f^* is induced from

Proposition

For any complex line bundle $L \to X$, there exists (up to homotopy) a unique map $f : X \to \mathbb{C}P^\infty$ such that $L \cong f^*(\mathcal{O}(1))$.

Quillen's Result

Question: In integral cohomology, for any line bundles L, K over space X, the first Chern class satisfies

$$c_1(L \otimes K) \cong c_1(L) + c_1(K).$$

What about other cohomology theories?

Theorem (Quillen)

Every complex oriented cohomology theory E determines a formal group law F_E over the ring $E^* = E^*(\mathrm{pt})$ by

$$c_1(L \otimes K) \cong F_E(c_1(L), c_1(K)).$$

Examples

- The integral cohomology $H\mathbb{Z}$ corresponds to

$$F_{H\mathbb{Z}}(x, y) = x + y.$$

- The complex K-theory KU corresponds to

$$F_{KU}(x, y) = x + y + \beta xy,$$

where $\beta \in \pi_*(KU)$ is the Bott element of degree 2.

Formal Group Laws in Char 0 Rings

Recall that for $F, G \in \operatorname{Fgl} R$, a morphism $f : F \to G$ is a power series $f \in R[[x]]$ such that

$$f(F(x,y)) = G(f(x), f(y)).$$

If f is invertible, then it is an isomorphism.
If R has characteristic 0, then $\log_F \in R[[x]]$ given by

$$\log_F(x) = \int \frac{dx}{\partial_y F(x,y)|_{y=0}}$$

satisfies

$$\log_F(F(x,y)) = \log_F(x) + \log_F(y).$$

Formal Group Laws in Char 0 Rings

One can show

$$\log_F(x) = 0 + x + \text{ higher order terms},$$

so it is invertible, and we have

Proposition

If R is a ring with characteristic 0, then any formal group law over R is isomorphic to $F(x, y) = x + y$.

Formal Group Laws in Char p Rings

If R has characteristic p, then the p-series of formal group laws $F \in \mathrm{Fgl}\, R$ must be in the form $[p]_F(x) = \sum_{n \geq 1} v_n x^{p^n}$. Hence there is a well-defined height notion:

Definition

For a formal group law $F \in \mathrm{Fgl}\,(R)$,

- F has height at least n if $v_i = 0$ for all $i < n$,
- F has height exactly n if $v_i = 0$ for all $i < n$ and $v_n \in R^{\times}$,
- F has height ∞ if $[p]_F(x) = 0$.

Proposition

If two formal group laws are isomorphic, then they have same height.

Examples

- $F(x,y) = x + y$ has p-series $[p]_F(x) = px = 0$, so it has height ∞.
- $F(x,y) = x + y + xy$ has p-series $[p]_F(x) = (1+x)^p - 1 = x^p$, so it has height 1.
- H_n has p-series $[p]_F(x) = x^{p^n}$, so it has height n.

Thickening and Deformation

We assume k to be a characteristic p field.

Definition

An *infinitesimal thickening* of a field k is an Artinian local ring A with a surjective map $\phi : A \to k$, with $\ker \phi$ being the maximal ideal $\mathfrak{m}$ of A. A morphism between two infinitesimal thickenings of k is a local ring homomorphism $f : A \to A'$ which commutes with their quotient maps. They form a category Art_k.

Definition

Let F_0 be a formal group law over k, and let A be an infinitesimal thickening of k. We say $F \in \mathrm{Fgl}\, A$ is a *deformation* of F_0 over A if $F \equiv F_0 \mod \mathfrak{m}$. An isomorphism of deformations is an isomorphism of formal group laws $f : F \to F'$ such that $f(x) \equiv x \mod \mathfrak{m}$. They form a groupoid $\mathrm{Def}_{F_0}(A)$.

The Lubin-Tate Theorem

Theorem (Lubin, Tate)

Let k be a perfect field of characteristic p, and let F_0 be a formal group law of height n over k. Then there exists a complete local Noetherian $W(k)$-algebra $E_0(F_0, k)$, isomorphic to $W(k)[[u_1, \cdots, u_{n-1}]]$, which pro-represents the functor $\mathrm{Def}_{F_0}(-) : \mathsf{Art}_k \to \mathsf{Gpd}$.
More specifically, there exists a universal deformation F_{univ} of F_0 over E_0, such that for every $A \in \mathsf{Art}_k$, the natural bijection

$$\mathrm{Spf}\,(E_0)(A) \cong \mathrm{Def}_{F_0}(A)$$

is given by $f \mapsto f_* F_{univ}$ for $f : E_0 \to A \in \mathrm{Spf}\,(E_0)(A)$.

$$\begin{aligned} &\mathrm{Spf}\,(E_0)(A) \\ &= \{ \text{ local ring maps } f : E_0 \to A \text{ with } \mathfrak{m}_{E_0}^n \subseteq \ker f \text{ for some } n \} \end{aligned}$$

The Morava E-Theory

We can form a graded ring $E_n = E_0[\beta^{\pm 1}]$ by adjoining a formal element β with degree 2. F_{univ} can be seen as a formal group law over E_n.

Proposition

The formal group law $F_{univ} \in \mathrm{Fgl}\, E_n$ is Landweber exact. So it gives a even-periodic homology theory E_{n*} by

$$E_{n*}(X) = E_n \otimes_{MU_*} MU_*(X).$$

By Brown Representability Theorem, there exists a spectrum E_n such that $\pi_*(E_n) = E_0[\beta^{\pm 1}]$, and such that $E_{n*}(X) \cong \pi_*(E_n \wedge X)$. Then we automatically get a cohomology theory

$$E_n^*(X) := [X, E_n]^*.$$

This is the Morava E-Theory.

Universality

The Morava E-theory E_n is constructed from the universal deformation of a height n formal group law.
It is the universal cohomology theory that detects phenomena at chromatic level n.

Theorem (Ravenel)

If X is a finite p-local spectrum, then X is a homotopy limit of its chromatic tower

$$\cdots \to L_{E_2}(X) \to L_{E_1}(X) \to L_{E_0}(X).$$

Further Readings

- Jacob Lurie, Chromatic Homotopy Theory, Lecture series (Harvard Math 252x, Spring 2010).
- Piotr Pstrgowski, Finite-Height Chromatic Homotopy Theory, Lecture notes (Harvard Math 252y, Spring 2021).

Thank You

Thank you for your patience!